§6.2 Orthogonal sets

A set of vectors $\{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$ is an <u>orthogonal set</u> if $v_i \bullet v_j = 0$ for all $i \neq j$, i.e. the vectors are all orthogonal to each other.

For example in $\mathbb{R}^3$ $\{v_1, v_2, v_3\}$ is an orthogonal set where

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

This follows as

$$v_1 \cdot v_2 = -3 + 2 + 1 = 0$$

$$v_1 \cdot v_3 = 3 + 4 - 7 = 0$$

$$v_2 \cdot v_3 = -1 + 8 - 7 = 0$$

...se sets are nice to work with and nice properties.

Theorem

If $\{v_1, \dots, v_m\}$ is an orthogonal set of nonzero vectors, then $\{v_1, \dots, v_m\}$ is linearly independent.

In particular $\{v_1, \dots, v_m\}$ is a basis for the subspace $\text{span}\{v_1, \dots, v_m\}$.

Proof

Suppose there are scalars $c_1, \dots, c_m$ such that

$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0$$

Then $v_1 \cdot (c_1 v_1 + c_2 v_2 + \dots + c_m v_m) = v_1 \cdot 0 = 0$

$$c_1 (v_1 \cdot v_1) + c_2 \underbrace{(v_1 \cdot v_2)}_{=0} + \dots + c_m \underbrace{(v_1 \cdot v_m)}_{=0} = 0$$

$$c_1 \underbrace{(v_1 \cdot v_1)}_{\neq 0 \text{ since } v_1 \neq 0} = 0$$

so $c_1 = 0$

Now repeat by dotting the original equation with $v_2, v_3, \dots, v_m$ to conclude $c_2, c_3, \dots, c_m = 0$ as well.

Definition

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W which is also an orthogonal set.

Example

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$ (the standard basis)

is an orthogonal basis for $\mathbb{R}^n$.

Theorem

Let $\mathcal{B} = \{v_1, \ldots, v_m\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For any y in W there is a unique linear combination

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

where $$c_i = \frac{y \cdot v_i}{v_i \cdot v_i} \qquad i = 1, 2, \ldots, m$$

Proof

Apply the same technique as in the previous proof.

Example

We saw $B = \left\{ \overbrace{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}^{v_1}, \overbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}^{v_2}, \overbrace{\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}}^{v_3} \right\}$ is an orthogonal set, hence it is actually a basis for $\mathbb{R}^3$.

(since $\dim \mathbb{R}^3 = 3$ and these are 3 linearly independant vectors).

Express $y = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ in terms of v_1, v_2, v_3.

Solution: If $y = c_1 v_1 + c_2 v_2 + c_3 v_3$ then

- $c_1 = \dfrac{y \cdot v_1}{v_1 \cdot v_1} = \dfrac{(9+4+5)}{9+1+1} = \dfrac{18}{11}$
- $c_2 = \dfrac{y \cdot v_2}{v_2 \cdot v_2} = \dfrac{(-3+8+5)}{1+4+1} = \dfrac{10}{6} = 5/3$
- $c_3 = \dfrac{y \cdot v_3}{v_3 \cdot v_3} = \dfrac{(3+16-35)}{(1+16+49)} = \dfrac{-16}{66} = -8/33$

SO $$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \boxed{(18/11)\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + (5/3)\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + (-8/33)\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}}$$

Exercise: Verify that these values c_1, c_2, c_3 are the same you get by solving the matrix equation

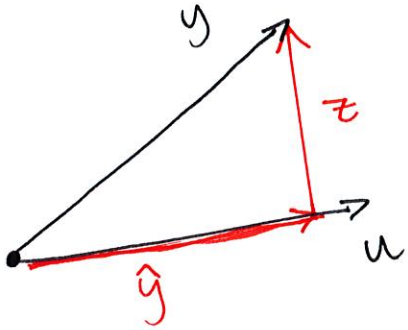
$$\begin{bmatrix} v_1 \,|\, v_2 \,|\, v_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = y$$

Orthogonal Projection

Suppose u is a vector in $\mathbb{R}^n$. Given any other vector y in $\mathbb{R}^n$, we aim to decompose y as

$$y = \hat{y} + z$$

where $\hat{y} = \alpha u$ for some scalar α and z is orthogonal to u



- $\hat{y}$ is called the <u>orthogonal projection</u> of y onto u.
- z is called the <u>component of y orthogonal</u> to u.

If we replace u by $L = \text{span}\{u\}$, the line along u, write $\hat{y} = \text{proj}_L y$ the projection of y onto line L.

Given y and u, how do we find $\hat{y}$ and z?

If $y = \hat{y} + z$, then $z = y - \hat{y} = y - \alpha u$ (since $\hat{y} = \alpha u$)

As z is orthogonal to u, we have

$$0 = z \cdot u = (y - \alpha u) \cdot u = y \cdot u - \alpha (u \cdot u)$$

Thus $\alpha = \dfrac{y \cdot u}{u \cdot u}$

So $\boxed{\hat{y} = \text{proj}_L y = \left(\dfrac{y \cdot u}{u \cdot u}\right) u}$

Moreover, notice the shortest distance between y and this line is $\text{dist}(y, \hat{y})$

$$= \|y - \hat{y}\|$$

$$= \|z\|$$

Example

Let $u = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

a) Find the orthogonal projection $\hat{y} = \text{proj}_L y$ where L is the line passing through u

b) Find the distance from y to L.

Solution

a) $\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u = \frac{12+3}{16+9} \cdot u = \frac{15}{25}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{3}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 12/5 \\ 9/5 \end{bmatrix}}$

b) $z = y - \hat{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 12/5 \\ 9/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$

$\|z\| = \sqrt{9/25 + 16/25} = \sqrt{25/25} = \boxed{1}$

Orthonormal Sets

- A set $\{v_1, \ldots, v_m\}$ is an orthonormal set if it is an orthogonal set of unit vectors
- An <u>orthonormal basis</u> of a subspace W of $\mathbb{R}^n$ is a basis which also an orthonormal set

Remark

An orthonormal set/basis can always be obtained from an orthogonal set/basis by normalizing each vector.

Theorem

An $m \times n$ matrix U has orthonormal columns if and only if $U^T \cdot U = I_n$

Definition

A square $n \times n$ matrix is orthogonal if A is invertible and $A^{-1} = A^T$

By the above a matrix is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.